Superreplication under Model Uncertainty in Discrete Time

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January 16, 2013

Abstract

We study the superreplication of contingent claims under model uncertainty in discrete time. We show that optimal superreplicating strategies exist in a general measure-theoretic setting; moreover, we characterize the minimal superreplication price as the supremum over all continuous linear pricing functionals on a suitable Banach space. The main ingredient is a closedness result for the set of claims which can be superreplicated from zero capital; its proof crucially relies on medial limits.

Keywords Knightian uncertainty; Nondominated model; Superreplication; Martingale measure; Medial limit; Hahn–Banach theorem

AMS 2000 Subject Classification 60G42; 91B28; 93E20

1 Introduction

We study the superreplication of contingent claims under model uncertainty in discrete-time financial markets. Model uncertainty is formalized by a set $\mathcal{P}$ of probability measures ("models") on a measurable space $(\Omega, \mathcal{F})$ and the superreplication is required to hold simultaneously under all measures $P \in \mathcal{P}$ ("$\mathcal{P}$-q.s."). More precisely, given an adapted process S and a random variable f, we are interested in determining the minimal superreplication price

$$x_*(f) = \inf \left\{ x \in \mathbb{R} : \exists H \text{ such that } x + H \bullet S_T \geq f \ \mathcal{P}\text{-q.s.} \right\};$$

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here H is a trading strategy (defined simultaneously under all $P \in \mathcal{P}$) and $H \bullet S_T$ is the terminal wealth resulting from trading in S at the discrete times $t = 1, \ldots, T$ according to H. Moreover, and this is our main goal, we want to show that an optimal superreplicating strategy exists; i.e., that the infimum is actually attained for some H.

Both problems are well understood in the absence of model uncertainty. Indeed, when $\mathcal{P}$ contains only one probability measure P, then an optimal strategy exists and

$$x_*(f) = \sup_{Q \in \mathcal{M}_e(P)} E_Q[f],$$

where $\mathcal{M}_e(P)$ denotes the set of all probability measures Q, equivalent to P, such that S is a Q-martingale. This holds under a no-arbitrage condition which by the fundamental theorem of asset pricing is equivalent to $\mathcal{M}_e(P)$ being nonempty, and embodies a fundamental duality between wealth processes and the linear pricing functionals $\{E_Q[\cdot], \, Q \in \mathcal{M}_e(P)\}$. It is well known that the price $x_*(f)$ can be relatively large for practical purposes; however, superreplication is of primal theoretical importance, for instance, in the solution of portfolio optimization problems. We refer to [4] and the references therein for the classical theory, and in particular to [14] for the discrete-time case.

Our aim is to obtain similar results in the situation of model uncertainty; i.e., when $\mathcal{P}$ can have many elements. If there exists a reference probability measure P_* with respect to which all $P \in \mathcal{P}$ are absolutely continuous, the resulting problem can be reduced to the classical one. We are interested in the case where this fails; i.e, $\mathcal{P}$ is *nondominated.* It is certainly reasonable to suppose that each of the possible models $P \in \mathcal{P}$ is viable in the usual sense and thus admits an equivalent martingale measure for S. As the superreplication problem depends only on the nullsets of the given measures, we may replace each $P \in \mathcal{P}$ by one of its equivalent martingale measures and as a result, we may assume directly that $\mathcal{P}$ itself consists of martingale measures.

For instance, we can take S to be the canonical process on the path space $\Omega = \mathbb{R}^T$ and let $\mathcal{P}$ be the set of all probabilities P such that P-a.s., S is positive and S_{t+1}/S_t is in a given interval I for all t; i.e., there is uncertainty about the log-increments of S, only the bound I is given. (A similar setup was used in [10] as a discrete approximation to the G-Brownian motion of [26].) More generally, given a process S on some measurable space, we can prescribe any collection $\mathcal{N}$ of sets and take $\mathcal{P}$ to be the family of all martingale measures not charging $\mathcal{N}$. In the special case where $\mathcal{N}$ is the collection of nullsets of a given reference measure P_*, this yields the classical set of absolutely continuous martingale measures, but generically, it yields a

nondominated set.

Our main result (Theorem 5.4) states that optimal superreplicating strategies exist in a general measure-theoretic setting. Moreover, $x_*(f)$ is described as the supremum over all continuous linear pricing functionals on a certain Banach space.

The main mathematical novelty in this paper is a closedness result (Theorem 4.1) for the cone $\mathfrak{C}$ of contingent claims which can be superreplicated from initial capital $x = 0$. A natural space for this result is introduced; namely, we consider the locally convex space $\mathfrak{L}^1$ of measurable functions with the seminorms given by $\{E_P[|\cdot|], \; P \in \mathcal{P}\}$. Our result states that $\mathfrak{C}$ is sequentially closed in $\mathfrak{L}^1$ (which is not a sequential space in general), and its proof makes crucial use of the so-called medial limits. These are measurable Banach limits which exist under certain set-theoretic axioms, such as the Continuum Hypothesis (see Section 2 for details). To have closedness rather than sequential closedness, we move to a Banach space $\mathbf{L}^1 \subseteq \mathfrak{L}^1$ whose topology, given by the norm $\|f\|_1 = \sup_{P \in \mathcal{P}} E_P[|f|]$, is stronger than the one of $\mathfrak{L}^1$. The main result is then obtained by a Hahn–Banach separation argument resembling the classical theory.

There are at least two obvious questions which are not answered in this paper. First, while $x_*(f)$ is described as the supremum over all continuous linear pricing functionals, we do not establish that (or when, rather) we have the formula $x_*(f) = \sup_{P \in \mathcal{P}} E_P[f]$. This question will be studied in future work; see also Remark 5.5. Second, we do not discuss the possible extension of the present results to the case of continuous-time processes with jumps.

To the best of our knowledge, there are no previous existence results for superreplication under model uncertainty in discrete time, and more generally for price processes S with jumps (except in the case where strategies are constants; cf. [28]). There are, however, results for continuous processes S with "volatility uncertainty" in specific setups; see [3, 13, 23, 25, 27, 29, 30, 31] and the references therein. All these results have been obtained by control-theoretic techniques which, as far as we have been able to see, cannot be applied in the presence of jumps. A duality result (without existence) for a specific topological setup in discrete time was obtained in [9], while [8] gave a comparable result for the continuous case.

A related topic is the so-called model-free pricing introduced by [12, 18], where superreplication is achieved by trading in the stock S and (statically) in a given set of options. On the dual side, this is related to the set of all martingale measures for S which are compatible with the prices of the given options, and this set can also be nondominated. A survey can be found in [19]; recent results are [1, 2] in discrete time and [11, 16, 17] in the

continuous case. Once more, we are not aware of general existence results in the case with jumps (but [2] contains a counterexample). It remains to be seen if our techniques can be applied to this problem. Finally, it is worth noting that in contrast to the present work, the mentioned papers consider topological setups and contingent claims which are functions of S alone.

The remainder of this paper is organized as follows. Section 2 states the necessary facts about medial limits; Section 3 introduces the space $\mathfrak{L}^1$; Section 4 contains the market model and the closedness result; Section 5 states the main superreplication result, and the concluding Section 6 provides a counterexample showing that $\mathfrak{L}^1$ is not sequential.

2   Medial Limits

In this section, we state some properties of Mokobodzki's medial limit (cf. [22] or [6, Nos. X.3.55–57]), whose use in the framework of model uncertainty was first introduced in [24]. In a nutshell, a medial limit is a Banach limit that preserves (universal) measurability and commutes with integration. Medial limits are usually constructed by a transfinite induction that uses the Continuum Hypothesis (the axiom that $\aleph_1 = \operatorname{card} \mathbb{R}$). In fact, it is known that medial limits exist under weaker hypotheses (e.g., Martin's Axiom, which is compatible with the negation of the Continuum Hypothesis), cf. [15, 538S], but not under ZFC alone [21]. In the remainder of the paper, we assume that medial limits exist[1]. Moreover, since there are then many medial limits, we choose one and denote it by $\lim \operatorname{med}$. It works as follows.

If $\{x_n\}_{n \geq 1}$ is a bounded sequence of real numbers, $\lim \operatorname{med} x_n$ is a number between $\liminf x_n$ and $\limsup x_n$, and if $\{f_n\}_{n \geq 1}$ is a uniformly bounded sequence of random variables on a measurable space $(\Omega, \mathcal{F})$, $f = \lim \operatorname{med} f_n$ is defined via $f(\omega) := \lim \operatorname{med} f_n(\omega)$. The first important property of the medial limit is that f is then universally measurable; i.e., measurable with respect to the σ-field

$$\mathcal{F}^* := \bigcap_{P \in M_1(\Omega, \mathcal{F})} \mathcal{F} \vee \mathcal{N}^P,$$

where $M_1(\Omega, \mathcal{F})$ is the set of all probability measures on $\mathcal{F}$ and $\mathcal{N}^P$ is the collection of P-nullsets. In particular, if $\mathcal{F}$ is universally complete (i.e., $\mathcal{F} = \mathcal{F}^*$), then the medial limit preserves $\mathcal{F}$-measurability. The second

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[1]In fact, we see little reason not to follow the advice of Dellacherie and Meyer [5] and "adopt the Continuum Hypothesis with the same standing as the Axiom of Choice."

important property is that lim med commutes with integration; that is,

$$\int (\operatorname{lim\,med} f_n)\, d\mu = \operatorname{lim\,med} \int f_n\, d\mu$$

whenever μ is a finite (possibly signed) measure on $(\Omega, \mathcal{F})$. We refer to [22, Theorem 2] for these results. The medial limit can be extended to nonnegative sequences $\{x_n\}_{n \geq 1}$ via

$$\operatorname{lim\,med} x_n := \sup_{m \in \mathbb{N}} \operatorname{lim\,med}(x_n \wedge m) \in [0, \infty].$$

If $\{x_n\}_{n \geq 1}$ is a general sequence, we set

$$\operatorname{lim\,med} x_n := \operatorname{lim\,med} x_n^+ - \operatorname{lim\,med} x_n^-$$

provided that the limits on the right-hand side are finite (or, equivalently, that $\operatorname{lim\,med} |x_n| < \infty$). The following properties are consequences of the fact that lim med commutes with integration; cf. the proofs of [22, Theorems 3,4].

Lemma 2.1. *Let $\{f_n\}_{n \geq 1}$ be a sequence of random variables on $(\Omega, \mathcal{F})$ and set*

$$f := \begin{cases} \operatorname{lim\,med} f_n & \text{if } \operatorname{lim\,med} |f_n| < \infty, \\ +\infty & \text{otherwise.} \end{cases}$$

Moreover, let μ be a finite signed measure on $(\Omega, \mathcal{F})$.

(i) *We have $\int |f|\, d|\mu| \leq \sup_n \int |f_n|\, d|\mu|$.*

(ii) *If $\{f_n\}_{n \geq 1}$ is μ-uniformly integrable, then f is μ-integrable and we have $\int f\, d\mu = \operatorname{lim\,med} \int f_n\, d\mu$.*

(iii) *If $\{f_n\}_{n \geq 1}$ converges in measure μ to some μ-a.e. finite random variable g, then $f = g$ μ-a.e.*

3   The Space $\mathfrak{L}^1$

Let $\mathcal{P}$ be a collection of probability measures on a measurable space $(\Omega, \mathcal{F})$. A subset $A \subseteq \Omega$ is called $\mathcal{P}$-*polar* if $A \subseteq A'$ for some $A' \in \mathcal{F}$ satisfying $P(A') = 0$ for all $P \in \mathcal{P}$ and a property is said to hold $\mathcal{P}$-*quasi surely* or $\mathcal{P}$-*q.s.* if it holds outside a $\mathcal{P}$-polar set. Consider the set of $\mathcal{F}$-measurable, real-valued functions on Ω and identify any two functions which coincide $\mathcal{P}$-q.s. We denote by $\mathfrak{L}^0 = \mathfrak{L}^0(\Omega, \mathcal{F}, \mathcal{P})$ the set of all such equivalence classes; in the sequel, we shall often not distinguish between these classes and actual functions.

Definition 3.1. The space $\mathfrak{L}^1(\Omega, \mathcal{F}, \mathcal{P})$ consists of all $f \in \mathfrak{L}^0(\Omega, \mathcal{F}, \mathcal{P})$ such that $\|f\|_{L^1(P)} := E_P[|f|] < \infty$ for all $P \in \mathcal{P}$. We equip $\mathfrak{L}^1(\Omega, \mathcal{F}, \mathcal{P})$ with the Hausdorff, locally convex vector topology induced by the family of seminorms $\{\| \cdot \|_{L^1(P)} : P \in \mathcal{P}\}$.

To wit, a net $\{f_\lambda\}$ in $\mathfrak{L}^1 = \mathfrak{L}^1(\Omega, \mathcal{F}, \mathcal{P})$ converges to $f \in \mathfrak{L}^1$ if and only if $E_P[|f_\lambda - f|] \to 0$ for all $P \in \mathcal{P}$; i.e., if convergence holds in each of the spaces $L^1(P)$. It is important that the closedness of a set in $\mathfrak{L}^1$ is *not determined by sequences* in general (cf. Example 6.1); i.e., we have to distinguish sequential closedness and topological closedness. This is at the heart of certain difficulties that we have encountered in our study; for instance, it is the reason why the problem mentioned in Remark 5.5(ii) is nontrivial.

The space $\mathfrak{L}^1(\Omega, \mathcal{F}, \mathcal{Q})$ is defined similarly when $\mathcal{Q}$ is a family of finite, possibly signed measures. In accordance with the usual notion of boundedness in a topological vector space, we shall say that a subset $\Theta \subseteq \mathfrak{L}^1(\Omega, \mathcal{F}, \mathcal{Q})$ is *bounded* if

$$\sup_{f \in \Theta} \|f\|_{L^1(Q)} < \infty \quad \text{for all} \quad Q \in \mathcal{Q}.$$

The following is easily deduced from Lemma 2.1.

Lemma 3.2. *Let $\mathcal{F}$ be universally complete and let $\{f_n\}_{n \geq 1}$ be a bounded sequence in $\mathfrak{L}^1(\Omega, \mathcal{F}, \mathcal{Q})$. Then*

$$\{\lim \operatorname{med} |f_n| = \infty\} \quad \text{is } \mathcal{Q}\text{-polar}$$

and $f := \lim \operatorname{med} f_n$ defines an element of $\mathfrak{L}^1(\Omega, \mathcal{F}, \mathcal{Q})$ satisfying

$$\|f\|_{L^1(Q)} \leq \sup_n \|f_n\|_{L^1(Q)} \quad \text{for all} \quad Q \in \mathcal{Q}.$$

Moreover, if $\{f_n\}_{n \geq 1}$ has a limit g in $\mathfrak{L}^1(\Omega, \mathcal{F}, \mathcal{Q})$, then $f = g$ $\mathcal{Q}$-q.s.

4 Sequential Closedness of $\mathfrak{C} \subseteq \mathfrak{L}^1$

Let $(\Omega, \mathcal{F})$ be a measurable space equipped with a filtration $(\mathcal{F}_t)_{t \in \{0,1,\ldots,T\}}$, where $T \in \mathbb{N}$. We shall assume throughout that

$$\mathcal{F}_t \text{ is universally complete, for all } t.$$

Moreover, let S be a scalar adapted process, the *stock price process*. We consider a nonempty set $\mathcal{P}$ of *martingales measures* for S; i.e., probability

measures under which S is a martingale. We shall denote by $\mathcal{H}$ the set of predictable processes, the *trading strategies*. Given $H \in \mathcal{H}$, the corresponding *wealth process* (from initial capital zero) is the discrete-time integral process

$$H \bullet S = (H \bullet S_t)_{t \in \{0,1,\dots,T\}}, \quad H \bullet S_t = \sum_{u=1}^{t} H_u \Delta S_u,$$

where $\Delta S_u = S_u - S_{u-1}$ is the price increment.

The main result of this section is that the cone $\mathfrak{C}$ of all claims which can be superreplicated from initial capital $x = 0$ is sequentially closed in $\mathfrak{L}^1 = \mathfrak{L}^1(\Omega, \mathcal{F}, \mathcal{P})$. We denote by $\mathfrak{L}^0_+$ the set of ($\mathcal{P}$-q.s.) nonnegative random variables.

Theorem 4.1. *Let $\mathcal{P} \neq \emptyset$ be a set of martingale measures for S and*

$$\mathfrak{C} := \big(\{H \bullet S_T : H \in \mathcal{H}\} - \mathfrak{L}^0_+\big) \cap \mathfrak{L}^1.$$

Then $\mathfrak{C}$ is sequentially closed in $\mathfrak{L}^1$.

Before stating the proof of the theorem, we show the following "compactness" property; it should be seen as a consequence of the "absence of arbitrage" which is implicit in our setup because $\mathcal{P}$ consists of martingale measures.

Lemma 4.2. *Let $\{W^n = H^n \bullet S_T - K^n\}_{n \geq 1} \subseteq \mathfrak{C}$ be a sequence which is bounded in $\mathfrak{L}^1$. Then for all $t \in \{1, \dots, T\}$,*

$$\{H^n_t \Delta S_t\}_{n \geq 1} \quad \text{is bounded in } \mathfrak{L}^1.$$

Proof. It suffices to show that for each $t \in \{1, \dots, T\}$,

$$\{H^n \bullet S_t\}_{n \geq 1} \quad \text{is bounded in } \mathfrak{L}^1. \tag{4.1}$$

Since $\{W^n\}_{n \geq 1}$ is bounded in $\mathfrak{L}^1$ and K^n is nonnegative, it follows immediately that $\{(H^n \bullet S_T)^-\}_{n \geq 1}$ is also bounded in $\mathfrak{L}^1$. Now fix n and $P \in \mathcal{P}$ and recall that P is a martingale measure for S. Therefore, the stochastic integral $H^n \bullet S$ is a local P-martingale, but since we already know that $E_P[(H^n \bullet S_T)^-] < \infty$, we even have that $H^n \bullet S$ is a true martingale; cf. [20, Theorems 1, 2]. As a result, $E_P[(H^n \bullet S_T)^+] = E_P[(H^n \bullet S_T)^-]$ for all n and P, and therefore, $\{(H^n \bullet S_T)^+\}_{n \geq 1}$ is bounded in $\mathfrak{L}^1$, like the sequence of negative parts. So far, we shown that

$$\{H^n \bullet S_T\}_{n \geq 1} \quad \text{is bounded in } \mathfrak{L}^1. \tag{4.2}$$

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To obtain the same statement for $t < T$, we note that for every $P \in \mathcal{P}$, the martingale property of $H^n \bullet S$ yields that

$$\|H^n \bullet S_t\|_{L^1(P)} = \|E_P[H^n \bullet S_T | \mathcal{F}_t]\|_{L^1(P)} \leq \|H^n \bullet S_T\|_{L^1(P)}$$

since the conditional expectation is a contraction on $L^1(P)$. Hence, (4.2) implies the claim (4.1). □

Proof of Theorem 4.1. Let $W^n = H^n \bullet S_T - K^n$ be a sequence in $\mathfrak{C}$ which converges to some $W \in \mathfrak{L}^1$; we need to find $H \in \mathcal{H}$ such that $W - H \bullet S_T \leq 0$ $\mathcal{P}$-q.s. Indeed, being convergent in $\mathfrak{L}^1$, the sequence $\{W^n\}_{n \geq 1}$ is necessarily bounded in $\mathfrak{L}^1$; hence, by Lemma 4.2, $\{H_t^n \Delta S_t\}_{n \geq 1}$ is bounded in $\mathfrak{L}^1$ for fixed $t \in \{1, \dots, T\}$. As S is a martingale and in particular integrable under each $P \in \mathcal{P}$, we can define the *finite* signed measures $Q_{t,P}$ by

$$dQ_{t,P}/dP = \Delta S_t.$$

Let $\mathcal{Q} = \{Q_{t,P}\}_{P \in \mathcal{P}}$, then the above means that the sequence $\{H_t^n\}_{n \geq 1}$ is bounded in $\mathfrak{L}^1(\Omega, \mathcal{F}_t, \mathcal{Q})$. Thus, Lemma 3.2 implies that $H_t := \lim \operatorname{med} H_t^n$ is finite $\mathcal{Q}$-q.s. Setting $H_t = 0$ on the set $\{\lim \operatorname{med} |H_t^n| = +\infty\} \in \mathcal{F}_t$, we obtain a process $H \in \mathcal{H}$. It remains to check that $K := H \bullet S_T - W$ is nonnegative $\mathcal{P}$-q.s. Indeed, since $W^n \to W$ in $\mathfrak{L}^1$, we know from Lemma 3.2 that $W = \lim \operatorname{med} W^n$ $\mathcal{P}$-q.s. In view of

$$H \bullet S_T = \sum_{t=1}^{T} (\lim \operatorname{med} H_t^n) \Delta S_t = \lim \operatorname{med} \sum_{t=1}^{T} H_t \Delta S_t = \lim \operatorname{med}(H^n \bullet S_T),$$

we conclude that

$$K = H \bullet S_T - W = \lim \operatorname{med}(H^n \bullet S_T - W_n) = \lim \operatorname{med} K^n.$$

As each K^n is nonnegative, the result follows. □

Remark 4.3. (i) The extension to multivariate price processes is not immediate. It requires a way to deal with "redundant assets" in our setting.

(ii) Presently, we do not know any general (and verifiable) sufficient conditions for $\mathfrak{C} \subseteq \mathfrak{L}^1$ to be topologically closed.

5   Main Result

In this section, we show that optimal superreplicating strategies exist and we characterize the minimal superreplication price in a dual way. As in the preceding section, $\mathcal{P}$ is a nonempty set of martingale measures for the scalar process S which is adapted to the universally complete filtration $(\mathcal{F}_t)$.

Definition 5.1. We introduce the normed vector space

$$\mathbf{L}^1 = \{f \in \mathfrak{L}^1 : \|f\|_1 < \infty\}, \quad \text{where} \quad \|f\|_1 = \sup_{P \in \mathcal{P}} E_P[|f|].$$

An element $f \in \mathbf{L}^1$ is $\mathcal{P}$-*uniformly integrable* if $\lim_{n \to \infty} \|f \mathbf{1}_{\{|f| \geq n\}}\|_1 = 0$.

We remark that if a nonnegative claim $f \in \mathfrak{L}^0$ can be superreplicated from some finite initial capital, then necessarily $f \in \mathbf{L}^1$. Moreover, one can check that the set of all $\mathcal{P}$-uniformly integrable functions coincides with the $\mathbf{L}^1$-closure of the set of bounded measurable functions, and that $f \in \mathbf{L}^1$ is $\mathcal{P}$-uniformly integrable as soon as there exists a superlinearly growing function ψ such that $\|\psi(f)\|_1 < \infty$; cf. the proofs of [7, Propositions 18, 28].

Definition 5.2. We introduce the cone

$$\mathbf{C} := \mathfrak{C} \cap \mathbf{L}^1 \equiv \big(\{H \bullet S_T : H \in \mathcal{H}\} - \mathfrak{L}^0_+\big) \cap \mathbf{L}^1,$$

as well as the set of continuous linear pricing functionals,

$$\Pi = \big\{\ell \in (\mathbf{L}^1)^* : \ell(\mathbf{C}) \subseteq \mathbb{R}_- \text{ and } \ell(1) = 1\big\}.$$

Note that Π is indeed the set of all continuous and linear pricing mechanisms which are consistent with obvious no-arbitrage considerations. As $\mathbf{C}$ contains the nonpositive elements of $\mathbf{L}^1$, we see that $\ell(\mathbf{C}) \subseteq \mathbb{R}_-$ implies that ℓ is positive; i.e., $\ell(f) \geq 0$ whenever $f \geq 0$ $\mathcal{P}$-q.s.

It is obvious that $\mathbf{L}^1 \subseteq \mathfrak{L}^1$ and that the topology of $\mathbf{L}^1$ is stronger than the one of $\mathfrak{L}^1$. Since sequential closedness and topological closedness are equivalent in a normed space (which is indeed the reason for moving from $\mathfrak{L}^1$ to $\mathbf{L}^1$), the following is then an immediate consequence of Theorem 4.1.

Corollary 5.3. *Let $\mathcal{P} \neq \emptyset$ be a set of martingale measures for S. Then $\mathbf{C}$ is closed in $\mathbf{L}^1$.*

The following is our main result: the optimal superreplicating strategy exists and the minimal superreplication price is given by the supremum over all linear prices.

Theorem 5.4. *Let $\mathcal{P} \neq \emptyset$ be a set of martingale measures for S and let $f \in \mathbf{L}^1$ be such that f^+ is $\mathcal{P}$-uniformly integrable. Then*

$$\sup_{\ell \in \Pi} \ell(f) = \inf \big\{x \in \mathbb{R} : \exists\, H \in \mathcal{H} \text{ such that } x + H \bullet S_T \geq f \text{ } \mathcal{P}\text{-q.s.}\big\} \quad (5.1)$$

and the infimum is attained whenever it is not equal to $+\infty$.

We emphasize that a priori, (5.1) is an identity in $(-\infty, \infty]$, with the usual convention $\inf \emptyset = +\infty$. (As $f \in \mathbf{L}^1$, the value $-\infty$ is clearly not possible for the left-hand side.)

Proof. We first show the inequality "$\leq$" in (5.1). To this end, let $x \in \mathbb{R}$ and $H \in \mathcal{H}$ be such that

$$x + H \bullet S_T \geq f \quad \mathcal{P}\text{-q.s.}$$

As $f \in \mathbf{L}^1$, this implies that $(H \bullet S_T)^- \leq (f - x)^- \in \mathbf{L}^1$. On the other hand, as in the proof of Lemma 4.2, we have $E_P[(H \bullet S_T)^+] = E_P[(H \bullet S_T)^-]$ for all $P \in \mathcal{P}$, and so we deduce that $(H \bullet S_T)^+ \in \mathbf{L}^1$ as well. As a result, we have that $H \bullet S_T \in \mathbf{C}$. Now let $\ell \in \Pi$; then positivity and the defining properties of Π yield that

$$\ell(f) \leq \ell(x + H \bullet S_T) = x + \ell(H \bullet S_T) \leq x,$$

which proves the desired inequality.

We turn to the inequality "$\geq$" in (5.1) and the existence of an optimal superreplicating strategy. Let $x := \sup_{\ell \in \Pi} \ell(f) \in (-\infty, \infty]$. If $x = +\infty$, nothing remains to be shown, so we may assume that x is finite and show that $f \in x + \mathbf{C}$ (which immediately yields both the inequality and the existence). Suppose for contradiction that $f \notin x + \mathbf{C}$. Since the convex cone $\mathbf{C}$ is closed in $\mathbf{L}^1$ by Corollary 5.3, the Hahn-Banach theorem yields a continuous functional $\ell : \mathbf{L}^1 \to \mathbb{R}$ such that

$$\sup_{W \in \mathbf{C}} \ell(W) < \ell(f - x) < \infty. \tag{5.2}$$

In fact, since $\mathbf{C}$ is a cone containing zero, $\sup_{W \in \mathbf{C}} \ell(W) < \infty$ implies that

$$\sup_{W \in \mathbf{C}} \ell(W) = 0 \tag{5.3}$$

and in particular (5.2) states that

$$\sup_{\ell \in \Pi} \ell(f) = x < \ell(f). \tag{5.4}$$

Of course, (5.3) shows that $\ell(\mathbf{C}) \subseteq \mathbb{R}_-$; in particular, ℓ is positive. Using that f^+ is uniformly integrable, we see that $f \wedge n \to f$ in $\mathbf{L}^1$ and hence

$$0 < \ell(f - x) = \lim_{n \to \infty} \ell((f - x) \wedge n) \leq \limsup_{n \to \infty} \ell(n).$$

This shows that $\ell(1) = n^{-1}\ell(n) > 0$. By a normalization, we may assume that $\ell(1) = 1$; but then $\ell \in \Pi$, which contradicts (5.4). $\qquad\square$

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Remark 5.5. (i) It is not hard to see that the theorem is indeed a genuine generalization of the classical superreplication duality mentioned in the Introduction (apart from our assumption that the σ-fields are universally complete).

(ii) It is interesting to find conditions guaranteeing that

$$\sup_{\ell \in \Pi} \ell(f) = \sup_{P \in \mathcal{P}} E_P[f],$$

which, together with (5.1), would yield an even closer analogue of the classical duality. A partial answer (for a special case) can already be found in [9]. In the general case, this question seems to be surprisingly difficult and will be addressed in future work.

6 A Counterexample

The subsequent example features a σ-convex set $\mathcal{P}$ of martingale measures for the trivial process $S \equiv 0$ and shows that a positive, sequentially continuous functional ℓ on $\mathcal{L}^1$ need not be continuous. (Note that $\mathfrak{C} = \mathcal{L}^1_-$ when $S \equiv 0$, so that positivity and $\ell(\mathfrak{C}) \subseteq \mathbb{R}_-$ are equivalent.) In particular, the nullspace of ℓ is then a sequentially closed set which is not topologically closed.

Example 6.1. Let $\Omega = [0, 1]$, let $\mathcal{F}$ be its Borel σ-field and let

$$\mathcal{P} = \left\{ \sum_{k \geq 1} \alpha_k \delta_{x_k} \,:\, \{x_k\}_{k \geq 1} \subseteq [0, 1],\ 0 \leq \alpha_k \leq 1,\ \sum_{k \geq 1} \alpha_k = 1 \right\}.$$

Then the Lebesgue measure μ induces a sequentially continuous functional on $\mathcal{L}^1(\Omega, \mathcal{F}, \mathcal{P})$ which is not topologically continuous.

Proof. Any $f \in \mathcal{L}^1$ is bounded, for otherwise there exist $x_n \in [0, 1]$ such that $|f(x_n)| \geq 2^n$ and therefore $E_P[|f|] = +\infty$ for $P := \sum_{n \geq 1} 2^{-n} \delta_{x_n}$, contradicting that $P \in \mathcal{P}$. Moreover, a sequence f_n in $\mathcal{L}^1$ converges to zero if and only if it is uniformly bounded and converges pointwise; i.e.,

$$\sup_{n \geq 1,\, x \in [0,1]} |f_n(x)| < \infty \qquad \text{and} \qquad f_n(x) \to 0, \quad x \in [0, 1]. \tag{6.1}$$

Indeed, (6.1) implies the convergence in $\mathcal{L}^1$ by the bounded convergence theorem (applied for each $P \in \mathcal{P}$). Conversely, let f_n converge to zero in $\mathcal{L}^1$, then the pointwise convergence must hold since $\delta_x \in \mathcal{P}$ for all $x \in [0, 1]$. Moreover, being convergent in $\mathcal{L}^1$, $\{f_n\}_{n \geq 1}$ must be bounded in $L^1(P)$ for

every $P \in \mathcal{P}$. If $\{f_n\}_{n \geq 1}$ is not uniformly bounded, then after passing to a subsequence, there exist $x_n \in [0, 1]$ such that $|f_n(x_n)| \geq n2^n$. Hence,

$$\sup_{n \geq 1} E_P[|f_n|] = +\infty \quad \text{for} \quad P := \sum_{n \geq 1} 2^{-n} \delta_{x_n},$$

which is again a contraction. Therefore, we have the characterization (6.1) for sequential convergence in $\mathfrak{L}^1$.

As a consequence, $\ell = E_\mu[\,\cdot\,]$ is a sequentially continuous linear functional on $\mathfrak{L}^1$ for any probability measure μ. However, when μ is the Lebesgue measure, it cannot be topologically continuous because otherwise $E_\mu[\,\cdot\,]$ would have to be dominated by finitely many of the seminorms $\{E_P[|\,\cdot\,|], P \in \mathcal{P}\}$, which is clearly not the case. $\qquad\square$

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